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A NOTE ON RANDOM TIMES

J.W. PITMAN and T.P. SPEED

Department of Probability and Statistics, University of Sheffield, Sheffield, UK

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Abstract. A generalisation of the notion of stopping time is stated, and related to similar generalisations introduced by Bahadur, Kemperman, Siegmund and others with a view to permitting auxiliary experimentation to enter into the definition of stopping rule. The main aim of this note is to draw attention to the conditional independence implicit in the definitions of these writers, and briefly indicate some consequences of this.

random time, conditional independence
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1. Introduction and description of results

Suppose that $(X_n,\ n = 1, 2, ...)$ is a sequence of random variables defined on a probability space $(\Omega, \mathcal{F}, \mathrm{P})$, and let $\mathcal{F}_n$ denote the σ-field generated by the random variables $X_1, X_2, ..., X_n$. In the theory of optional stopping of such a process (X_n), the random times considered are commonly assumed to be *stopping times of* $(\mathcal{F}_n)$, that is to say, extended positive integer-valued random variables t such that for each $n = 1, 2, ...$, the event $\{t > n\}$ lies in the σ-field $\mathcal{F}_n$ determined by the evolution of the process up to time n. Several authors have also considered stopping procedures involving the outcomes of random experiments auxiliary to the basic process (X_n); in this connection we mention Bahadur [2], Kemperman [6], Singh [9], Siegmund [8], Chow, Robbins and Siegmund [4], and Arjas and Speed [1].

In order to provide a unified approach to the work of these authors we make the definition which follows. Let $(\Omega, \mathcal{F}, \mathrm{P})$ be a probability space and let us refer to an extended positive integer-valued random variable defined on Ω as a *random time*. Suppose that $(\mathcal{F}_n,\ n = 1, 2, ...)$ is an increasing sequence of sub-σ-fields of $\mathcal{F}$, and let $\mathcal{F}_\infty$ denote the smallest σ-field containing every $\mathcal{F}_n$, $n = 1, 2, ...$.

Definition 1.1. A random time t is a *randomised stopping time* of ($\mathcal{F}_n$) if for each $n = 1, 2, \ldots$, the event $\{t > n\}$ and the σ-field $\mathcal{F}_\infty$ are conditionally independent given $\mathcal{F}_n$.

The point of this note is to state Propositions 2.4 and 2.5 below which utilise some elementary properties of conditional independence to give various equivalent formulations of the above definition. These formulations show that the kinds of random times considered by Bahadur and Siegmund are essentially the same and just randomised stopping times according to the above definition, and it is clear that the random times considered by Kemperman and Singh are also included. We do not discuss here any applications of randomised stopping times, but refer the reader to the papers and books mentioned above.

When $\mathcal{F}_n$ is the σ-field generated by random variables $X_1, \ldots, X_n$, we refer to a (randomised) stopping time of ($\mathcal{F}_n$) as a *(randomised) stopping time of* (X_n). It will be seen that a randomised stopping time of (X_n) can be thought of as being generated in the following way: an observer watches the evolution of the process (X_n) as time n increases, until a random time t when he stops observing the process; if at time k he has not yet stopped observing the process, the observer notes the value of X_k and then decides according to the outcome of some random experiment whether to stop at time k or to continue to observe the process. The random time t is a randomised stopping time of (X_n) if for each k, the outcome of the random experiment at time k and the as yet unobserved future $(X_n,\ k < n < \infty)$ are conditionally independent given the observed past $(X_n,\ 1 \le n \le k)$. The random time t is a stopping time of (X_n) if for each k, the decision at time k is made deterministically (and measurably) according to the past $(X_n,\ 1 \le n \le k)$.

A consequence of Proposition 2.5 is that properties of randomised stopping times associated with Markov processes or martingales can be immediately deduced from the well-known properties of stopping times of these processes. Indeed, let $(X_n,\ n = 1, 2, \ldots)$ be a sequence of random variables adapted to an increasing sequence of σ-fields ($\mathcal{F}_n$, $n = 1, 2, \ldots$), and suppose that (X_n) is a Markov process (respectively, martingale) with respect to ($\mathcal{F}_n$). If t is a randomised stopping time of ($\mathcal{F}_n$) and $\mathcal{F}_n^t$ denotes the σ-field generated by $\mathcal{F}_n$ and the events $\{t = 1\}, \ldots$ $\ldots, \{t = n\}$, then it follows from the equivalence of (i) and (iv) in Proposition 2.5 that (X_n) is also a Markov process (respectively, martingale) with respect to ($\mathcal{F}_n^t$), and since t is a stopping time of ($\mathcal{F}_n^t$), all the standard results for stopping times of Markov processes and martingales can be applied at once to randomised stopping times of these processes.

For the sake of simplicity, we have only considered here random times associated with processes whose time set is the positive integers, but most of the discussion is easily adapted to the other usual time sets.

2. Details

Let $\mathbf{N}$ denote the set of natural numbers $\{1, 2, \ldots, n, \ldots\}$. Suppose throughout that $(\Omega, \mathcal{F}, \mathbf{P})$ is a probability space and that $(\mathcal{F}_n, n \in \mathbf{N})$ is an increasing sequence of sub-σ-fields of $\mathcal{F}$, with $\mathcal{F}_\infty$ the smallest sub-σ-field of $\mathcal{F}$ containing each $\mathcal{F}_n$. The reader is referred to [7] for a treatment of conditional independence.

Remark 2.1. Recalling from the introduction the definition of a randomised stopping time of $(\mathcal{F}_n)$, we observe that alternative but equivalent definitions are obtained by replacing the set $\{t > n\}$ apearing in the definition by any one of the sets $\{t \leq n\}$, $\{t = n\}$ and $\{t \neq n\}$.

Examples 2.2. Any random time independent of $\mathcal{F}_\infty$ is a randomised stopping time of $(\mathcal{F}_n)$, and so too is any stopping time of $(\mathcal{F}_n)$. For a less trivial example, consider a real-valued process (X_n) defined on $(\Omega, \mathcal{F}, \mathbf{P})$ and suppose that Y is a real-valued random variable independent of the process (X_n). Let $t = \inf\{n : X_n \geq Y\}$. Then it is easily seen that t is a randomised stopping time of (X_n). A similarly defined randomised stopping time of a continuous time process finds an application in [3, p. 276].

For another example, suppose that (X_n) is a Markov chain and let T_n be the time of the n^{th} visit to state i. Let T be any stopping time of (X_n) and define a random time t by $t = \inf\{n : T_n \geq T\}$, so that $t - 1$ is the number of visits to state i before time T. Then t is a randomised stopping time of (T_n), as may be seen from the fact that $\{t > n\} = \{T_n < T\}$, [7, IV T 41] and the strong Markov property (cf. [5, p. 27, proof of Theorem (76)]).

For sub-σ-fields $\mathcal{A}$ and $\mathcal{B}$ of $\mathcal{F}$, let us denote by $\mathcal{A} \vee \mathcal{B}$ the smallest sub-σ-field of $\mathcal{F}$ which contains both $\mathcal{A}$ and $\mathcal{B}$. Suppose we are given sub-σ-fields $\mathcal{E}_1$, $\mathcal{E}_2$ and $\mathcal{E}$ of $\mathcal{F}$.

Lemma 2.3. *The following statements are equivalent:*

(i) $\mathcal{E}_1$ *and* $\mathcal{E}_2$ *are conditionally independent given* $\mathcal{E}$;
(ii) $\mathbf{P}[A \mid \mathcal{E} \vee \mathcal{E}_2] = \mathbf{P}[A \mid \mathcal{E}]$ *a.s. for every set* A *in* $\mathcal{E}_1$;
(iii) $\mathcal{E} \vee \mathcal{E}_1$ *and* $\mathcal{E} \vee \mathcal{E}_2$ *are conditionally independent given* $\mathcal{E}$;
(iv) $\mathbf{E}\{Y \mid \mathcal{E} \vee \mathcal{E}_2\} = \mathbf{E}\{Y \mid \mathcal{E}\}$ *a.s. for every integrable* $\mathcal{E} \vee \mathcal{E}_1$*-measurable random variable* Y.

In (ii) and (iii) the subscripts 1 and 2 can be interchanged to give further statements equivalent to (i).

Proof. The equivalence of (i) and (ii) is proved in [7, II T 51]. The further equivalence of (iii) and (iv) follows by repeated application of this result.

With the aid of Lemma 2.3, the conditional independence condition in the definition of randomised stopping time can now be rephrased in a multitude of ways. Proposition 2.4 below displays some minimal conditions for a random time to be a randomised stopping time of $(\mathcal{F}_n)$, while in Proposition 2.5 the conditional independence is exploited to the full to give some strong properties of randomised stopping times.

Suppose that t is a random time on $(\Omega, \mathcal{F}, \mathbf{P})$.

Proposition 2.4. *The following statements are equivalent*:
(i) t *is a randomised stopping time of* $(\mathcal{F}_n)$;
(ii) *for all* $n \in \mathbf{N}$, $\mathbf{P}[t > n \mid \mathcal{F}_\infty] = \mathbf{P}[t > n \mid \mathcal{F}_n]$ *a.s.*;
(iii) *for all* $n \in \mathbf{N}$, $A \in \mathcal{F}_\infty$, $\mathbf{P}[t > n, A] = \int_{\{t>n\}} \mathbf{P}[A \mid \mathcal{F}_n]\, d\mathbf{P}$.
Further statements equivalent to (i) *are obtained by replacing the set* $\{t > n\}$ *appearing in* (ii) *and* (iii) *by any of* $\{t \leq n\}$, $\{t = n\}$ *and* $\{t \neq n\}$.

Proof. Let $\mathcal{S}_n$ denote the sub-σ-field of $\mathcal{F}$ generated by the event $\{t > n\}$. By definition, t is a randomised stopping time of $(\mathcal{F}_n)$ if and only if for each n in $\mathbf{N}$, the σ-fields $\mathcal{S}_n$ and $\mathcal{F}_\infty$ are conditionally independent given $\mathcal{F}_n$. The equivalence of (i), (ii) and (iii) now follows from the equivalence of (i) and (ii) in Lemma 2.3 since $\mathcal{F}_\infty \vee \mathcal{F}_n = \mathcal{F}_\infty$ and $\mathcal{S}_n \vee \mathcal{F}_n$ has an obvious simple structure. Using Remark 2.1 the remaining assertions can be proved in an identical manner.

Continuing to suppose that t is a random time on $(\Omega, \mathcal{F}, \mathbf{P})$, let $\mathcal{F}'_n$ denote the smallest sub-σ-field of $\mathcal{F}$ containing $\mathcal{F}_n$ and the events $\{t = 1\}, \ldots, \{t = n\}$.

Proposition 2.5. *The following statements are equivalent:*

(i) *t is a randomised stopping time of* $(\mathcal{F}_n)$;

(ii) *for each $n \in \mathrm{N}$ the σ-fields $\mathcal{F}_n^t$ and $\mathcal{F}_\infty$ are conditionally independent given* $\mathcal{F}_n$;

(iii) *for each* $n \in \mathrm{N}$, $\mathrm{E}\{Y \mid \mathcal{F}_\infty\} = \mathrm{E}\{Y \mid \mathcal{F}_n\}$ *a.s. for each integrable $\mathcal{F}_n^t$-measurable random variable* Y;

(iv) *for each* $n \in \mathrm{N}$, $\mathrm{E}\{Z \mid \mathcal{F}_n^t\} = \mathrm{E}\{Z \mid \mathcal{F}_n\}$ *a.s. for each integrable $\mathcal{F}_\infty$-measurable random variable Z.*

Proof. Let $\mathcal{F}_n$ denote the sub-σ-field of $\mathcal{F}$ generated by the events $\{t = 1\}, \ldots, \{t = n\}$, so that $\mathcal{F}_n^t = \mathcal{F}_n \vee \mathcal{F}_n$. The fact that $(\mathcal{F}_n)$ is an increasing sequence of σ-fields ensures that t is a randomised stopping time of $(\mathcal{F}_n)$ if and only if for each n in N, the σ-fields $\mathcal{F}_n$ and $\mathcal{F}_\infty$ are conditionally independent given $\mathcal{F}_n$, and the proposition now follows by applying Lemma 2.3.

Put another way, the equivalence of (i) and (ii) in Proposition 2.5 means that t is a randomised stopping time of $(\mathcal{F}_n)$ if and only if there exists an increasing sequence of σ-fields $(\mathcal{G}_n)$ within $\mathcal{F}$ such that $\mathcal{F}_n \subseteq \mathcal{G}_n$, t is a stopping time of $(\mathcal{G}_n)$, and $\mathcal{G}_n$ and $\mathcal{F}_\infty$ are conditionally independent given $\mathcal{F}_n$. With this conditional independence criterion written in the form

$$\mathrm{P}[A \mid \mathcal{G}_n] = \mathrm{P}[A \mid \mathcal{F}_n] \qquad \text{a.s.}$$

for all $n \in \mathrm{N}$ and $A \in \mathcal{F}_\infty$, this is just the property required by Siegmund of his 'randomised stopping variables for $(\mathcal{F}_n)$".

Given an increasing sequence $(\mathcal{F}_n)$ of σ-fields in a probability space $(\Omega, \mathcal{F}, \mathrm{P})$, it may be that the probability space as it stands is not large enough to support many randomised stopping times of $(\mathcal{F}_n)$. As an extreme case, if $\mathcal{F} = \mathcal{F}_\infty$, then the only randomised stopping times of $(\mathcal{F}_n)$ are stopping times of $(\mathcal{F}_n)$. For this reason, it seems reasonable to consider the possibility of enlarging the original probability space in some way to allow room for experimentation auxiliary to $\mathcal{F}_\infty$. Consider, for example, the following procedure used by Bahadur [2]. Suppose that there is given for each n, an $\mathcal{F}_n$-measurable function a_n with $0 \leq a_n \leq 1$.

Observe the sequence of σ-fields $(\mathcal{F}_n)$ in succession, and given that the first m σ-fields have been observed, conduct an auxiliary random experiment with probability of success equal to the observed value of a_m, stopping at the time of the first success. This procedure will define a random time t on a probability space $(\Omega', \mathcal{F}', \mathbf{P}')$ constructed from $(\Omega, \mathcal{F}, \mathbf{P})$ and all necessary auxiliary experiments. This probability space will contain an isomorphic image of $\mathcal{F}$ in $\mathcal{F}'$ on which $\mathbf{P}'$ agrees with $\mathbf{P}$, and after identifying σ-fields and random variables defined on $(\Omega, \mathcal{F}, \mathbf{P})$ with their isomorphic copies in $(\Omega', \mathcal{F}', \mathbf{P}')$ it is being assumed that for each $n \in \mathbf{N}$, the event $\{t > n\}$ and the σ-field $\mathcal{F}_\infty$ are conditionally independent given $\mathcal{F}_n$ and $\{t \geq n\}$, and that

$$\mathbf{P}'[\{t = n\} \mid \mathcal{F}_n, \{t \geq n\}] = a_n \qquad \text{on } \{t \geq n\}\,.$$

The construction of the probability space $(\Omega', \mathcal{F}', \mathbf{P}')$ and random time t is easily formalised, and it may be shown that t is a randomised stopping time of $(\mathcal{F}_n)$ in $(\Omega', \mathcal{F}', \mathbf{P}')$, with

$$\mathbf{P}'[\{t > n\} \mid \mathcal{F}_\infty] = (1 - a_1) \dots (1 - a_n)\,, \qquad n \in \mathbf{N}\,.$$

Moreover, if t^* is any randomised stopping time of $(\mathcal{F}_n)$ defined (in the obvious way) on an enlargement $(\Omega^*, \mathcal{F}^*, \mathbf{P}^*)$ of $(\Omega, \mathcal{F}, \mathbf{P})$, then a randomised stopping time of $(\mathcal{F}_n)$ having the same joint distribution with $\mathcal{F}_\infty$ as t^* can be constructed in the manner described above by taking

$$a_n = \mathbf{P}^*[t = n \mid \mathcal{F}_n] / \mathbf{P}^*[t \geq n \mid \mathcal{F}_n]\,.$$

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